

The squashed fuzzy sphere, fuzzy strings and the Landau problem

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Abstract

We discuss the squashed fuzzy sphere, which is a projection of the fuzzy sphere onto the equatorial plane, and use it to illustrate the stringy aspects of noncommutative field theory. We elaborate explicitly how strings linking its two coincident sheets arise in terms of fuzzy spherical harmonics. In the large N limit, the matrix-model Laplacian is shown to correctly reproduce the semi-classical dynamics of these charged strings, as given by the Landau problem.

1 Introduction

Field theory on noncommutative (NC) spaces has been studied intensively from various points of view in the past decades. One of the original motivations was the (naive) hope that the UV-divergences of quantum field theory would be regularized on a noncommutative space, due to the presence of an intrinsic noncommutative scale Λ_{NC} . This hope turned out not to be vindicated. Rather, NC field theory behaves very differently from ordinary field theory at scales far above Λ_{NC} , where the basic degrees of freedom display a string-like or dipole-like nature. This is already implicit in the trivial observation that NC fields are matrices or operators, which thus have two indices, and are naturally represented in t'Hooft's double line notation [1]. Indeed, scalar fields on a noncommutative space arise in string theory as open strings starting and ending on a D-brane with B field [2, 3]. This suggests a dipole-like nature of noncommutative fields [4, 5], which is also implicit in the matrix-model realization of noncommutative gauge theory and its relation with string theory [6], culminating in the remarkable proposals [7, 8] that string theory might be *defined* in terms of matrix models. In particular, the IKKT matrix model is tantamount to noncommutative $\mathcal{N} = 4$ SYM on \mathbb{R}_θ^4 .

In the same vein, the interactions determined by the algebra of noncommutative scalar fields with momentum far above Λ_{NC} is also very different from the commutative case; this

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can be seen easily on the quantum plane \mathbb{R}_θ^2 , but also e.g. on the fuzzy sphere [9]. Since all these high-scale modes are probed in QFT via loop contributions, it should not be surprising that NC quantum field theory (NCQFT) is typically quite different from ordinary QFT, and seems consistent only for very special models³. The stringy nature of NCQFT manifests itself also in the gravitational aspects of noncommutative gauge theory [11–14] and the notorious UV/IR mixing [15].

These insights are very useful also to study NC field theory per se, without any direct relation with string theory. It allows to understand better its intrinsic properties, and suggests a different organization of its fundamental degrees of freedom. In the present paper, we provide a particularly simple and explicit illustration of the stringy nature of noncommutative scalar fields, in the example of noncommutative scalar field theory on the squashed fuzzy sphere PS_N^2 . This is a noncommutative space obtained by projecting the fuzzy sphere S_N^2 on the equatorial plane. It should be viewed as a stack of two coinciding fuzzy disks with opposite (non-constant) Poisson structures, glued together at their boundary. The dipole or string picture discussed above suggests that there should be string-like modes connecting these two sheets, with opposite charges at the ends moving in the fields B_+ and B_- on the two sheets. Here $B_+ = -B_-$ corresponds to the symplectic forms i.e. the inverse Poisson structures on the two sheets. At low energies, these should behave like point-like charged objects moving in an effective magnetic field $B_+ - B_-$, which – focusing on the center of the disks in a suitable scaling limit – should reduce to the Landau problem⁴.

With this in mind, we study free scalar field theory on PS_N^2 , and identify the low-energy modes and their effective action. We can indeed identify the lowest eigenmodes of the (matrix) Laplacian with string-like modes connecting the opposite sheets, which reproduce precisely the energy levels and degeneracies of the Landau problem. They are identified as fuzzy spherical harmonics \hat{Y}_m^l with large quantum numbers $m \approx \pm l$. For the lowest Landau level, the modes at (or near) the origin can be expressed succinctly in terms of coherent states localized at the origin of the two sheets, thus exhibiting their stringy nature. This is also related to recent results on the low-energy modes of coinciding or intersecting branes on squashed $SU(3)$ branes [18].

The present paper hence demonstrates how an appropriate organization of the degrees of freedom⁵ can illuminate the stringy physics hidden in NCFT, which transcends the picture of conventional field theory.

2 The Landau levels

We recall the quantum mechanical description of a (spinless) charged particle moving perpendicular to an uniform magnetic field along the z -axis,

$$\vec{B} = B\hat{e}_z.$$

³This includes the maximally supersymmetric $\mathcal{N} = 4$ Super-Yang-Mills, which is nothing but the IKKT model, and a particular matrix model interpreted as scalar field theory [10].

⁴For a treatment of the Landau problem on the fuzzy sphere with monopole charge see [16, 17]. This is not directly related to the problem under investigation here.

⁵A somewhat related organization of fields in terms of the so-called the matrix base was used in [19] to analyze perturbation theory for scalar field theory on the quantum plane.

The Hamiltonian for such a set-up is

$$H = \frac{1}{2\mu} \left(\vec{P} - \frac{q}{c} \vec{A} \right)^2 \quad (2.1)$$

where \vec{A} is the vector potential related to the magnetic field, which has the form

$$\vec{A} = \frac{B}{2} \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix} \quad (2.2)$$

in the Landau gauge. Inserting (2.2) into (2.1) and introducing the cyclotron (or Larmor) frequency

$$\omega_c = \frac{-qB}{\mu c} \quad (2.3)$$

the Hamiltonian can be written as

$$H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{\mu\omega_c^2}{8} (X^2 + Y^2) + \frac{\omega_c}{2} L_z = H_{xy} + \frac{\omega_c}{2} L_z,$$

where L_z is the angular momentum operator in z -direction, and H_{xy} is the Hamiltonian of a two dimensional harmonic oscillator with frequency $\frac{\omega_c}{2}$. We can reformulate the problem in terms of the ladder operators

$$\begin{aligned} a_r &= \frac{1}{2} \left(\beta (X - iY) + \frac{i}{\beta\hbar} (P_x - iP_y) \right), \\ a_l &= \frac{1}{2} \left(\beta (X + iY) + \frac{i}{\beta\hbar} (P_x + iP_y) \right), \end{aligned}$$

with $\beta = \sqrt{\frac{\mu\omega_c}{2\hbar}}$. These are the annihilation operators of right and left circular quanta respectively. We introduce the number operators

$$\begin{aligned} N_r &= a_r^\dagger a_r, \\ N_l &= a_l^\dagger a_l \end{aligned}$$

so that

$$\begin{aligned} H_{xy} &= (N_r + N_l + 1) \frac{\hbar\omega_c}{2}, \\ L_z &= (N_r - N_l) \hbar. \end{aligned}$$

Now it is evident that a_r^\dagger (a_l^\dagger) create right (left) circular quanta. Both raise the energy by $\frac{\hbar\omega_c}{2}$, but acting with a_r^\dagger increases the additional angular momentum by \hbar , while acting with a_l^\dagger decreases the angular momentum by \hbar . Thus the Hamiltonian (2.1) has the form

$$H = \left(N_r + \frac{1}{2} \right) \hbar\omega_c$$

with eigenvalues

$$E = \left(n_r + \frac{1}{2} \right) \hbar\omega_c \quad (2.4)$$

and eigenfunctions

$$|\chi_{n_r, n_l}\rangle = \frac{1}{\sqrt{n_r! n_l!}} (a_r^\dagger)^{n_r} (a_l^\dagger)^{n_l} |\chi_{0,0}\rangle, \quad n_l, n_r \in \mathbb{N}.$$

Note that the energy depends only on n_r but is independent of n_l , thus the energy states corresponding to a particular Landau level n_r are infinitely degenerate. It is not hard to see that the wave-functions for the lowest Landau level $n_r = 0$ are concentric circles around the origin with radius measured by n_l ,

$$\chi_{0, n_l}(\rho, \varphi) = \frac{\beta}{\sqrt{\pi n_l!}} e^{-i n_l \varphi} (\beta \rho)^{n_l} e^{-\frac{\beta^2 \rho^2}{2}} \quad (2.5)$$

in polar coordinates.

If we include spin, the Hamiltonian is modified as follows

$$H = \left(N_r + \frac{1}{2} - \frac{\sigma_z}{2} g \right) \hbar \omega_c \quad (2.6)$$

where σ_z is the spin operator in z -direction and g the g -factor dependent on the type of particle.

3 The fuzzy sphere S_N^2

The fuzzy sphere S_N^2 [20, 21] is a quantization of the usual sphere S^2 with a cutoff in angular momentum, which contains N quanta of area. The quantization of S^2 is given by a quantization map \mathcal{Q} ,

$$\begin{aligned} \mathcal{Q}: \quad \mathcal{C}_n(S^2) &\rightarrow \mathcal{M}_N = \text{Mat}(N, \mathbb{C}) \\ x^a &\mapsto X^a = \kappa J^a \end{aligned} \quad (3.1)$$

which maps in particular the embedding functions x^a on S^2 to quantized embedding functions $X^a = \kappa J^a$ on S_N^2 . Here J^a are the generators of $\mathfrak{su}(2)$ in the $N = 2n+1$ -dimensional irreducible representation, $\mathcal{C}_n(S^2)$ is the space of polynomials on S^2 of degree $\leq n$ and \mathcal{M}_N is the algebra of complex $N \times N$ matrices. Since the quadratic Casimir operator has the form

$$\vec{J}^2 = C_N \mathbb{1} \quad \text{with} \quad C_N = \frac{1}{4} (N^2 - 1),$$

the radial constraint of a sphere with radius r

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = r^2$$

is recovered if we set

$$\kappa^2 = \frac{r^2}{C_N}.$$

We introduce a constant which is the analogue of \hbar

$$\hbar = \kappa r = \frac{r^2}{\sqrt{C_N}} \quad (3.2)$$

and the commutative limit is given by $\hbar \rightarrow 0$ as $N \rightarrow \infty$ for fixed radius. The generators X^a of the algebra \mathcal{M}_N satisfy the commutation relations

$$[X^a, X^b] = i\hbar C^{ab}_c X^c =: i\Theta^{ab}, \quad (3.3)$$

$$C^{abc} = r^{-1} \varepsilon^{abc}, \quad (3.4)$$

$$(\Theta^{ab})_{S_N^2} = \frac{\hbar}{r} \begin{pmatrix} 0 & X^3 & -X^2 \\ -X^3 & 0 & X^1 \\ X^2 & -X^1 & 0 \end{pmatrix}. \quad (3.5)$$

To complete the definition of the quantization map \mathcal{Q} , we decompose \mathcal{M}_N into irreducible representations under the adjoint action of $\mathfrak{su}(2)$

$$\begin{aligned} \mathcal{M}_N \cong (N) \otimes (\bar{N}) &= (1) \oplus (3) \oplus \dots \oplus (2N-1) \\ &= \left\{ \hat{Y}_0^0 \right\} \oplus \dots \oplus \left\{ \hat{Y}_m^{N-1} \right\}. \end{aligned} \quad (3.6)$$

This defines the fuzzy spherical harmonics \hat{Y}_m^l , and allows to write down a natural definition for the quantization map \mathcal{Q} for polynomial functions of degree less than or equal to $n = 2N+1$:

$$\begin{aligned} \mathcal{Q}: \mathcal{C}_n(S^2) &\rightarrow \mathcal{M}_N = \text{Mat}(N, \mathbb{C}) \\ Y_m^l &\mapsto \hat{Y}_m^l, \end{aligned}$$

compatible with the $SO(3)$ symmetry. Here Y_m^l are the usual spherical harmonics. In the limit $N \rightarrow \infty$, we recover the full algebra of polynomial functions on S^2 .

The commutation relations (3.5) define a quantization of the Poisson structure

$$\begin{aligned} \{x^a, x^b\} &= \hbar C^{ab}_c x^c =: \theta^{ab}, \\ C^{abc} &= r^{-1} \varepsilon^{abc} \\ \theta^{ab} &= \frac{\hbar}{r} \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix} \end{aligned}$$

which corresponds to the $SO(3)$ -invariant symplectic 2-form

$$\omega_N = \frac{1}{\hbar} C_{abc} x^a dx^b \wedge dx^c \quad (3.7)$$

and satisfies the flux quantization condition $2\pi N = \int_{S^2} \omega_N$. Thus the fuzzy sphere S_N^2 is the quantization of the symplectic manifold (S^2, ω_N) . Furthermore, the Laplace operator on the fuzzy sphere is defined by

$$\square = \frac{1}{\hbar^2} \sum_{a=1}^3 [X^a, [X^a, \cdot]]. \quad (3.8)$$

This type of matrix Laplacian arises naturally in the context of Yang-Mills models⁶ [14].

⁶For example in the IKKT model [8], the matrices X^a transform in the adjoint of some $U(N)$ gauge group. Assuming that they acquire non-trivial expectation values such as $X^a \sim J^a$ (3.2), the $U(N)$ gauge symmetry is spontaneously broken, and linearized transversal fluctuations on such a background are governed by the Laplace operator (3.8). This can be viewed as a variant of the Higgs mechanism. Here, we simply take (3.8) as a natural starting point, ignoring possible extra degrees of freedom which may arise in other contexts.

3.1 Fuzzy spherical harmonics

The fuzzy spherical harmonics \hat{Y}_m^l were identified in equation (3.6) as the irreducible representations of $SU(2)$ acting on the non-commutative algebra \mathcal{M}_N , analogous to the commutative case up to a cutoff. It is easy to see that they are also eigenfunctions of the Laplace operator

$$\square \hat{Y}_m^l = \frac{\kappa^2}{\hbar^2} l(l+1) \hat{Y}_m^l = \frac{1}{r^2} l(l+1) \hat{Y}_m^l, \quad (3.9)$$

in analogy to the classical case, with the same $2l+1$ -fold degeneracy. We can get more information on the explicit (matrix) form of the \hat{Y}_m^l for fixed N using the representation theory of $SU(2)$. Consider a basis where the Cartan generator H of $SU(2)$ is diagonal. Since m gives the eigenvalue of H , all the matrices \hat{Y}_m^l are diagonal, \hat{Y}_1^l have entries only along the first diagonal above the main diagonal, \hat{Y}_2^l have entries only along the second diagonal above the main diagonal and so forth. An analogous statement can be made for \hat{Y}_{-1}^l , \hat{Y}_{-2}^l , etc. below the main diagonal. The entries of the matrices are symmetric w.r.t. the anti-diagonal, and their values are decreasing with increasing distance from the anti-diagonal. Clearly the maximal value for l is $l_{max} = N-1$, and all matrices with $|m| > l$ vanish.

4 The squashed fuzzy sphere PS_N^2

In this section we discuss the squashed fuzzy sphere, which is interpreted as projection of the fuzzy sphere onto the equatorial plane [18]. This arises e.g. as building block of cosmological solutions in the IR-regulated IKKT matrix model [22]. In particular, we explain how strings linking its two coincident sheets arise in terms of noncommutative functions. The relation of matrix models with noncommutative gauge theory is illustrated by showing how the description of these strings in noncommutative field theory reproduces the semi-classical dynamics of these charged strings as given by the Landau problem.

A projection Π of a classical sphere onto its equatorial plane⁷ is achieved simply by replacing the three embedding functions $x^a : S^2 \hookrightarrow \mathbb{R}^3$ by only two embedding functions x^1 and x^2 , dropping x^3 :

$$\begin{array}{ccc} S^2 & \rightarrow & \mathbb{R}^3 \\ p & \mapsto & x^a(p) \end{array} \quad \xrightarrow{\Pi} \quad \begin{array}{ccc} \mathbb{R}^2 & & \\ & \mapsto & x^a(p), \quad a = 1, 2 \end{array} \quad (4.1)$$

Here we keep the same space of functions on S^2 , but change the embedding information given by the x^a . After projecting, the two hemispheres are stacked one onto another as two coinciding disks glued at the boundary.

Accordingly, we define the projected or squashed fuzzy sphere PS_N^2 in terms of the *two* generators X^a , $a = 1, 2$. They generate the same algebra of fuzzy functions $\text{Mat}(N, \mathbb{C})$ as for S_N^2 , but will lead to a different fuzzy Laplacian. It can be viewed as two projected fuzzy disks glued at the boundary. The relation between the fuzzy disk and the fuzzy sphere can be seen explicitly by expressing X^3 in terms of the two independent generators X^1, X^2 :

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = r^2 \quad \Rightarrow \quad (X^3)^2 = r^2 - (X^1)^2 - (X^2)^2. \quad (4.2)$$

⁷Note that we are considering an orthogonal projection rather than a stereographic projection here.

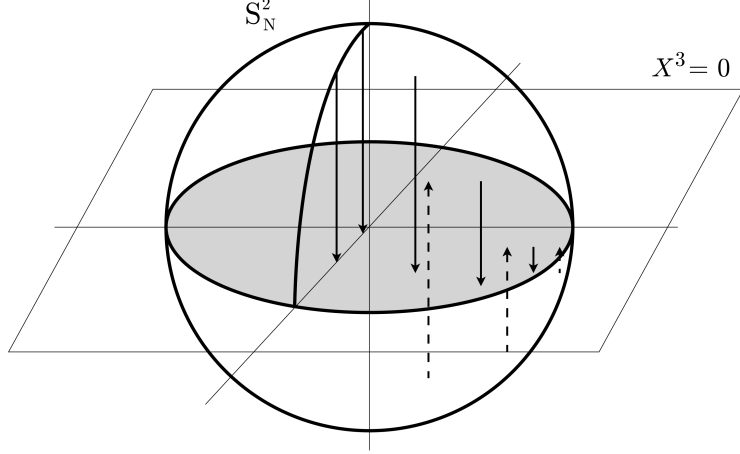


Figure 1: A schematic depiction of the orthographic projection Π of the fuzzy sphere onto the $X^3 = 0$ plane. The solid arrows indicate projections of the upper hemisphere, while the dashed arrows are projections from the lower hemisphere. The shaded area is the area onto which is projected, i.e. the interleaved fuzzy disks.

We define

$$X_{\pm}^3 = \pm \sqrt{r^2 - (X^1)^2 - (X^2)^2} \quad (4.3)$$

as positive respectively negative part of X^3 . Then X_{\pm}^3 reduces in the semi-classical (i.e. Poisson) limit to the embedding functions x_{\pm}^3 of upper respectively lower hemisphere in \mathbb{R}^3 . Then the matrix Laplacian on the squashed fuzzy sphere is

$$\square_S = \frac{1}{\hbar^2} \sum_{i=1}^2 [X^i, [X^i, \cdot]] . \quad (4.4)$$

4.1 Poisson structure

The commutators of the generators X^1, X^2 of the squashed fuzzy sphere define in the semi-classical limit a Poisson structure on the projected disks. This is nothing but the push-forward of the Poisson structure on S^2 by Π . On the upper sheet, we have

$$\{x^1, x^2\} = \frac{\hbar}{r} x_+^3(x^1, x^2) = \frac{\hbar}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2} = \theta_+^{12} \quad (4.5)$$

while on the lower sheet we have

$$\{x^1, x^2\} = \frac{\hbar}{r} x_-^3(x^1, x^2) = -\frac{\hbar}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2} = \theta_-^{12} \quad (4.6)$$

Thus the Poisson tensor on the two sheets indicated by \pm is given by

$$\theta_{\pm}^{ij} = \pm \frac{\sqrt{r^2 - (x^1)^2 - (x^2)^2}}{r} \hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (4.7)$$

We observe that the two coinciding fuzzy disks have opposite Poisson structure

$$-\theta_+^{ij} = \theta_-^{ij}, \quad (4.8)$$

and θ_{\pm} vanishes as we approach the edge, so that we have a smooth transition

$$r^2 - (x^1)^2 - (x^2)^2 = 0 \quad \Rightarrow \quad \theta_+^{ij} = \theta_-^{ij} = 0. \quad (4.9)$$

Of course the semi-classical treatment at the edge may be questioned, however this will not be important below. We observe that both the Poisson structure and the Laplacian are quite different from the corresponding structures on the single fuzzy disks defined in [23, 24] via a truncation of the quantum plane.

4.2 Effective gauge fields

In this section, we will obtain an interpretation of the matrix Laplacian \square in terms of non-commutative gauge theory. This will allow to identify particular functions on the squashed fuzzy sphere as charged strings linking its two sheets, and provide an explicit relation with the energy levels of Landau problem.

To understand this relation, we recall that gauge fields on the Moyal-Weyl quantum plane \mathbb{R}_{θ}^2 can be introduced as deformations

$$X^i = \bar{X}^i - \bar{\theta}^{ij} A_j(\bar{X}) \quad (4.10)$$

of the generators \bar{X}^i of \mathbb{R}_{θ}^2 , which satisfy

$$[\bar{X}^i, \bar{X}^j] = i\bar{\theta}^{ij} = i\bar{k} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.11)$$

The X^i are known as covariant coordinates [25]. Their commutators are given by

$$\begin{aligned} [X^i, X^j] &= [\bar{X}^i, \bar{X}^j] - \bar{\theta}^{jj'} [\bar{X}^i, A_{j'}] + \bar{\theta}^{ii'} [\bar{X}^j, A_{i'}] + \bar{\theta}^{ii'} \bar{\theta}^{jj'} [A_{i'}, A_{j'}] \\ &= -i\bar{\theta}^{ii'} \bar{\theta}^{jj'} (\bar{\theta}_{i'j'}^{-1} - \partial_{i'} A_{j'} + \partial_{j'} A_{i'} + i[A_{i'}, A_{j'}]) \\ &= i\bar{\theta}^{ij} - i\bar{\theta}^{ii'} \bar{\theta}^{jj'} F_{i'j'} \end{aligned} \quad (4.12)$$

where F_{ij} can be interpreted as field strength of the $U(1)$ gauge field⁸ A_i on \mathbb{R}_{θ}^2 . The commutators

$$\begin{aligned} [X^i, \phi] &= [\bar{X}^i, \phi] - \bar{\theta}^{ii'} [A_{i'}, \phi] \\ &= i\bar{\theta}^{ii'} (\partial_{i'} + i[A_{i'}, \phi]) \\ &= i\bar{\theta}^{ii'} D_{i'} \phi \end{aligned} \quad (4.13)$$

defines the covariant derivatives of a scalar field ϕ . In the semi-classical limit, the Poisson-brackets of $x^i \sim X^i$ can be expressed accordingly

$$\begin{aligned} \{x^i, x^j\} &= \{\bar{x}^i, \bar{x}^j\} - \bar{\theta}^{jj'} \{\bar{x}^i, A_{j'}\} + \bar{\theta}^{ii'} \{\bar{x}^j, A_{i'}\} + \bar{\theta}^{ii'} \bar{\theta}^{jj'} \{A_{i'}, A_{j'}\} \\ &= \bar{\theta}^{ij} - \bar{\theta}^{ii'} \bar{\theta}^{jj'} F_{i'j'} \\ &= \theta^{ij} \end{aligned} \quad (4.14)$$

⁸Recall that in noncommutative field theory, the field strength contains commutators even for abelian i.e. $U(1)$ gauge fields. However these terms are subleading in the semi-classical limit, and will be dropped here.

as deformation of the constant Poisson bracket $\{\bar{x}^i, \bar{x}^j\} = \bar{\theta}^{ij}$ by the field strength F_{ij}

$$F_{ij} = \partial_i A_j - \partial_j A_i - \{A_i, A_j\}.$$

Thus \bar{x}^i can be viewed as Darboux coordinates on $(\mathbb{R}^2, \{.,.\})$. The semi-classical version of (4.10)

$$x^i = \bar{x}^i - \bar{\theta}^{ij} A_j(\bar{x}) \quad (4.15)$$

therefore allows to interpret the difference between the x^i and the Darboux coordinates \bar{x}^i in terms of a $U(1)$ gauge field.

We now apply these insights to the example of the squashed fuzzy sphere. Since its finite-dimensional setting cannot strictly be viewed as a deformation of the quantum plane \mathbb{R}_θ^2 , we restrict ourselves to the semi-classical (i.e. Poisson) limit. More precisely, we consider the limit corresponding to $N \rightarrow \infty$, keeping the leading order in the noncommutativity scale \hbar . Higher powers in \hbar can be neglected as long as the physical momenta are sufficiently low.

As we have seen in (4.8), the squashed fuzzy sphere decomposes into an upper and a lower fuzzy disk, which arise by restricting the matrices to the upper and lower blocks defined by the positive and negative spectrum of X^3 :

$$X^i = \begin{pmatrix} X_+^i & 0 \\ 0 & X_-^i \end{pmatrix} \sim \begin{pmatrix} x_+^i & 0 \\ 0 & x_-^i \end{pmatrix} = \begin{pmatrix} \bar{x}^i - \bar{\theta}^{ij} A_j^+ & 0 \\ 0 & \bar{x}^i - \bar{\theta}^{ij} A_j^- \end{pmatrix} \quad (4.16)$$

with $+$ ($-$) indicating the upper (lower) sheet. Note that although the full Poisson structures $\theta_+^{ij}, \theta_-^{ij}$ have opposite sign, the Darboux coordinates \bar{x}^i define the *same* constant $\bar{\theta}^{ij}$ on the upper and the lower sheet,

$$\{\bar{x}^i, \bar{x}^j\} = \bar{\theta}^{ij} = \hbar \varepsilon^{ij}. \quad (4.17)$$

This is essential for an interpretation in terms of noncommutative gauge theory on a stack of coinciding branes.

Now we want to find the corresponding gauge fields A_i^\pm on the two sheets explicitly. From (4.7) we get

$$\{x_\pm^i, x_\pm^j\} = \theta_\pm^{ij} = \pm \frac{\sqrt{r^2 - (x^1)^2 - (x^2)^2}}{r} \hbar \varepsilon^{ij}$$

which indeed reduces to (4.17) for $x^1, x^2 \simeq \vec{0}$, and vanishes at the edge of the disk. We can rewrite equation (4.14) as

$$\bar{\theta}_{ii'}^{-1} \bar{\theta}_{jj'}^{-1} \theta^{ij} = -\bar{\theta}_{i'j'}^{-1} - F_{i'j'}$$

with

$$\bar{\theta}_{ij}^{-1} = \frac{1}{\hbar} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and obtain

$$\bar{\theta}_{ii'}^{-1} \bar{\theta}_{jj'}^{-1} \theta^{i'j'} = \frac{1}{\hbar r} \begin{pmatrix} 0 & \sqrt{r^2 - (x^1)^2 - (x^2)^2} \\ -\sqrt{r^2 - (x^1)^2 - (x^2)^2} & 0 \end{pmatrix}.$$

We thus obtain the field strength F on the different sheets as

$$F = \begin{pmatrix} F^+ & 0 \\ 0 & F^- \end{pmatrix}$$

with

$$F_{ij}^{\pm} = \frac{1}{k} \begin{pmatrix} 0 & 1 \mp \frac{1}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2} \\ -\left(1 \mp \frac{1}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2}\right) & 0 \end{pmatrix}.$$

To obtain explicit expressions for the gauge fields \vec{A}^{\pm} , we have to solve the following differential equations

$$F_{12}^{\pm} = k^{-1} \left(1 \mp \frac{1}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2} \right) = \partial_1 A_2^{\pm} - \partial_2 A_1^{\pm} - \{A_1^{\pm}, A_2^{\pm}\}. \quad (4.18)$$

Since $\{A_1^{\pm}, A_2^{\pm}\}$ is of higher order in k than $\partial_1 A_2^{\pm} - \partial_2 A_1^{\pm}$, it is negligible in the semi-classical limit, and (4.18) simplifies to

$$F_{12}^{\pm} \simeq \partial_1 A_2^{\pm} - \partial_2 A_1^{\pm}. \quad (4.19)$$

The solutions of this differential equations are given by

$$\vec{A}^{\pm}(\vec{x}) = \frac{1}{2k} \begin{pmatrix} -x^2 \pm K^1 \\ x^1 \pm K^2 \end{pmatrix} \quad (4.20)$$

where

$$\begin{aligned} K^1 &= \frac{1}{2r} \left(x^2 \sqrt{r^2 - (x^1)^2 - (x^2)^2} + (r^2 - (x^1)^2) \arctan \left(\frac{x^2}{\sqrt{r^2 - (x^1)^2 - (x^2)^2}} \right) \right), \\ K^2 &= \frac{-1}{2r} \left(x^1 \sqrt{r^2 - (x^1)^2 - (x^2)^2} + (r^2 - (x^2)^2) \arctan \left(\frac{x^1}{\sqrt{r^2 - (x^1)^2 - (x^2)^2}} \right) \right). \end{aligned}$$

It is easy to verify

$$F_{12}^{\pm} = \vec{\nabla} \times \vec{A}^{\pm}.$$

Now consider in more detail the covariant derivative (4.13) acting on general noncommutative scalar fields on the squashed fuzzy sphere including off-diagonal components,

$$\phi \equiv \Upsilon = \begin{pmatrix} \Upsilon_+ & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_- \end{pmatrix}. \quad (4.21)$$

We denote the scalar fields on PS_N^2 with Υ henceforth, to emphasize their stringy nature. Here Υ_{\pm} correspond to functions on the upper and lower sheet, respectively, while $\Upsilon_{12}, \Upsilon_{21}$ are naturally interpreted as strings connecting these sheets⁹. Then the covariant derivatives acting on the string-like modes is

$$\begin{aligned} D_i \Upsilon_{12} &= -i \bar{\theta}_{ii'}^{-1} [X^{i'}, \Upsilon_{12}] \sim \partial_i \Upsilon_{12} - i (A_i^+ - A_i^-) \Upsilon_{12} \\ D_i \Upsilon_{21} &= -i \bar{\theta}_{ii'}^{-1} [X^{i'}, \Upsilon_{21}] \sim \partial_i \Upsilon_{21} + i (A_i^+ - A_i^-) \Upsilon_{21} \end{aligned}$$

with

$$\vec{A}^+ - \vec{A}^- = k^{-1} \begin{pmatrix} K^1 \\ K^2 \end{pmatrix}.$$

⁹Equivalently, one may consider Υ as $\mathfrak{u}(2)$ -valued noncommutative gauge field on a single sheet.

We note that the off-diagonal string-like modes couple to the difference $\vec{A}^+ - \vec{A}^-$ of the gauge fields on the two sheets, and behave like charged objects moving in a background with field strength $F^+ - F^-$. In particular, the Laplacian (4.4) acting on these fields becomes

$$\square_S \Upsilon_{12} = \delta^{ij} D_i D_j \Upsilon_{12} \quad (4.22)$$

in the semi-classical limit. This is precisely the Hamiltonian for a charged particle moving in a magnetic field, as studied in section 2. We therefore expect that in the pole limit i.e. near the origin $\vec{x} = 0$ for $r \rightarrow \infty$, its spectrum should reproduce that of the Landau problem. This will be elaborated below.

Pole limit $\vec{x} = \vec{0}$. Near the pole we can expand the field strength in a Taylor series in x^i . Neglecting terms suppressed by $\mathcal{O}((\frac{x^i}{r})^2)$, we obtain

$$F_{12}^+ = 0, \quad F_{12}^- = \frac{2}{\tilde{k}}$$

and the gauge fields obtained from (4.22) are

$$\vec{A}^+ - \vec{A}^- = \frac{1}{\tilde{k}} \begin{pmatrix} x^2 \\ -x^1 \end{pmatrix}.$$

In particular, $(\vec{A}^+ - \vec{A}^-)$ acting on Υ_{21} corresponds to the field strength

$$F^+ - F^- = \frac{-2}{\tilde{k}} \quad (4.23)$$

while $(\vec{A}^- - \vec{A}^+)$ acting on Υ_{12} corresponds to the opposite field strength. Not surprisingly, the two sheets reduce for $r \rightarrow \infty$ to Moyal-Weyl quantum planes \mathbb{R}_θ^2 , with a constant field $F = \frac{2}{\tilde{k}}$ on the lower sheet. We can of course absorb the field strength in any given quantum plane by redefining $\hat{\theta}^{ij}$, but the difference between the two sheets is unambiguous.

Edge limit $(x^1)^2 + (x^2)^2 = r^2$. At the edge, the field strength F_{ij} becomes

$$F_{12}^\pm = \frac{1}{\tilde{k}}$$

on both sheets, consistent with the fact that the Poisson structures on the upper and lower sheet (4.9) have a smooth transition.

We observe that F_{12} is divergent as $\tilde{k} \rightarrow 0$. However this is not a problem, since we are interested in the physics for fixed \tilde{k} corresponding to fixed magnetic field B (4.26). In other words, we consider the limit corresponding to $N \rightarrow \infty$ while keeping B or \tilde{k} fixed.

4.3 Fuzzy Laplacian and its eigenfunction

In the previous chapter we arrived at an interpretation of the matrix Laplacian \square_S on the squashed fuzzy sphere in the semi-classical limit (4.22). Now we return to the fuzzy case, and

study this matrix Laplacian exactly. Comparing it with the Laplacian on the fuzzy sphere (3.8), we can write

$$\square_S = \square - \frac{1}{k^2} [X^3, [X^3, \cdot]]$$

In fact we can immediately write down all the eigenvectors and eigenvalues: they are given by the same fuzzy spherical harmonics \hat{Y}_m^l which diagonalize \square , since X^3 is proportional to J^3 and $J^3 \hat{Y}_m^l = m \hat{Y}_m^l$. Thus

$$\square_S \hat{Y}_m^l = \frac{1}{r^2} (l(l+1) - m^2) \hat{Y}_m^l.$$

Note that the spectrum of \square_S is independent of the matrix dimension N , up to the cutoff.

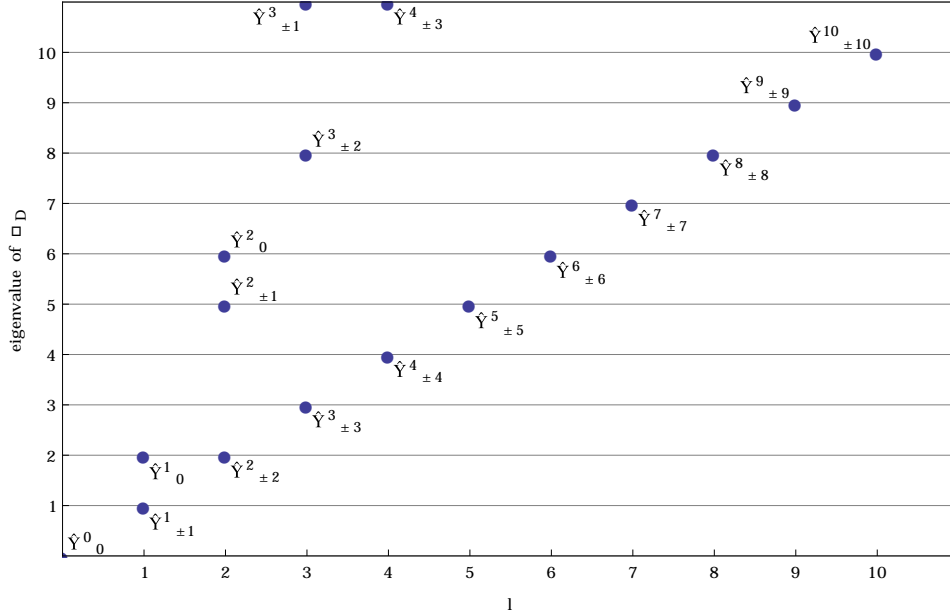


Figure 2: The first few eigenvalues of \square_S corresponding to \hat{Y}_m^l . Each eigenvalue where $m \neq 0$ is at least twice degenerated, since \hat{Y}_m^l and \hat{Y}_{-m}^l have the same eigenvalues. However, this does not mean that the $m = 0$ the eigenvalue is not degenerate, since e.g. \hat{Y}_0^2 and $\hat{Y}_{\pm 6}^6$ have the same eigenvalue given by 6.

Since the eigenvalues of \square_S are not independent of m anymore (unlike in \square), the degeneracy of each eigenvalue has a more complicated structure than for \square . Figure 2 shows the lowest eigenvalues and the corresponding states \hat{Y}_m^l .

Thus the \hat{Y}_m^l span the Hilbert space on the squashed fuzzy sphere. However to identify the string modes Υ_{12} , we need to identify those \hat{Y}_m^l , which have entries exclusively in the upper right block, because of equation (4.21). In chapter 3.1 we saw that with larger m , the entries of the \hat{Y}_m^l are farther away from the main diagonal, thus for $l \simeq l_{max}$ and $|m| \simeq l$ the \hat{Y}_m^l have entries solely in these string domains. Therefore for $m > 0$ the $\hat{Y}_{m \simeq l}^l$ serve as a basis for the Υ_{12} , when $m < 0$ for Υ_{21} . These are the string modes we are looking for.

4.4 Semi-classical limit and string states

Having identified the basis for the strings Υ_{12} (and Υ_{21}) as $\hat{Y}_{m \simeq l}^l$ (and $\hat{Y}_{m \simeq -l}^l$), we want to understand their precise relation with the states of the Landau problem, and relate the

spectrum of \square_S for these string states in the semi-classical limit.

Recall from chapter 3.1 that the quantum numbers for the fuzzy spherical harmonics for fixed matrix dimension N are given by

$$\begin{aligned} l &= 0, 1, \dots, l_{max}, & l_{max} &= N - 1, \\ m &= l, l - 1, \dots, -l. \end{aligned}$$

The distribution of the eigenfunctions in figure 2 already suggests which grouping of the \hat{Y}_m^l might be appropriate. The \hat{Y}_m^l with fixed difference $l - m$ lie on certain lines, as illustrated in figure 3. In order to appropriately describe the \hat{Y}_m^l states with $l \simeq l_{max}$ and $|m| \simeq l$ for large N , we define two new (small) quantum numbers L and M as the complement of l and m (which are large). Let us distinguish the two cases where m is positive and negative, since they are correlated to different strings, Υ_{12} and Υ_{21} respectively. Thus we define

$$\begin{aligned} L &:= l_{max} - l \in \{0, 1, 2, \dots\} \\ M &:= l - m \in \{0, 1, 2, \dots\} \quad (\text{for } m > 0), \\ M' &:= l + m \in \{0, 1, 2, \dots\} \quad (\text{for } m < 0). \end{aligned}$$

Since \square_S in (4.22) has the form of a squared momentum operator $(\partial + A)^2$, we multiply it with a factor of $\frac{1}{2\mu}$ in order to relate \square_S with the Hamiltonian (2.1), where μ is the mass of the particle. Then the eigenvalue equation for \square_S becomes

$$\frac{1}{2\mu} \square_S \hat{Y}_m^l = \frac{(l(l+1) - m^2)}{2\mu r^2} \hat{Y}_m^l.$$

We can now rewrite this in terms of our new quantum numbers L and M , and get

$$\begin{aligned} l(l+1) - m^2 &\stackrel{m \geq 0}{=} ((N - L - 1)(N - L) - (N - 1 - L - M)^2) \\ &= (-1 - L - 2M - 2LM - M^2 + (1 + 2M)N) \\ &\stackrel{-1-L-2M-2LM-M^2 \ll N}{=} 2N \left(M + \frac{1}{2} \right) + \mathcal{O}(1) \end{aligned}$$

for $m > 0$, and

$$l(l+1) - m^2 \stackrel{m \leq 0}{=} 2N \left(M' + \frac{1}{2} \right) + \mathcal{O}(1),$$

for $m < 0$. Here we assume that N is very large while M, L are small, as appropriate for the flat (pole) limit. Accordingly, we define a new basis of string modes as follows:

$$\begin{aligned} \Upsilon_{(12)}^{L,M} &= \hat{Y}_{l_{max}-L-M}^{l_{max}-L}, & l_{max} - L - M &> \frac{N}{2} \\ \Upsilon_{(21)}^{L,M'} &= \hat{Y}_{-(l_{max}-L)+M'}^{l_{max}-L}, & -l_{max} + L + M' &< -\frac{N}{2} \end{aligned} \quad (4.24)$$

Thus

$$\begin{aligned} \frac{1}{2\mu} \square_S \Upsilon_{(12)}^{L,M} &= \frac{N}{\mu r^2} \left(M + \frac{1}{2} \right) \Upsilon_{(12)}^{L,M} & \text{for } L = 0, 1, 2, \dots \\ \frac{1}{2\mu} \square_S \Upsilon_{(21)}^{L,M'} &= \frac{N}{\mu r^2} \left(M' + \frac{1}{2} \right) \Upsilon_{(21)}^{L,M'} & \text{for } L = 0, 1, 2, \dots \end{aligned} \quad (4.25)$$

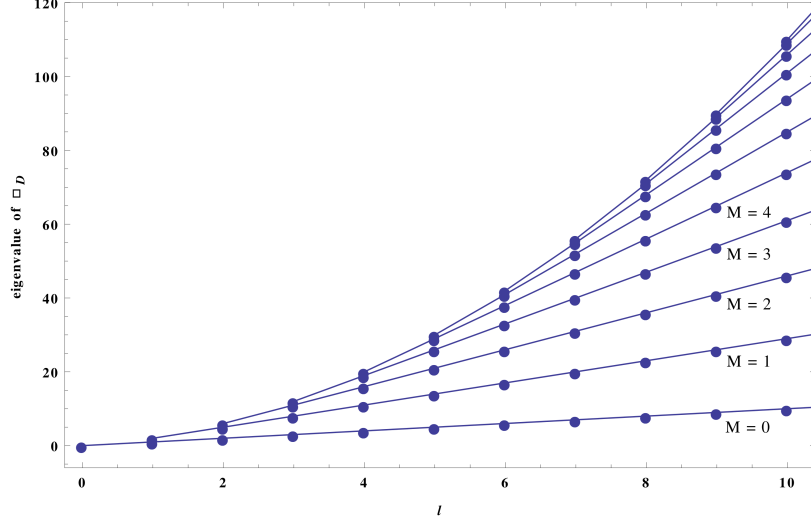


Figure 3: Levels $M = 0, \dots, 10$ for small l_{max} , if N is small. If N is large this domain is where l is small, i.e. these \hat{Y}_m^l are not suitable as basis for the strings.

We can compare this to the eigenvalue equation (2.4) of the Landau problem in chapter 2

$$\begin{aligned} H |\chi_{n_r, n_l}\rangle &= \hbar\omega_c \left(n_r + \frac{1}{2} \right) |\chi_{n_r, n_l}\rangle & \text{for } q < 0 & \quad \text{with } n_l = 0, 1, 2, \dots \\ H |\chi_{n_r, n_l}\rangle &= \hbar\omega_c \left(n_l + \frac{1}{2} \right) |\chi_{n_r, n_l}\rangle & \text{for } q > 0 & \quad \text{with } n_r = 0, 1, 2, \dots \end{aligned}$$

Note that we have both charged sectors $q = \pm 1$ realized at the same time, by the $\Upsilon_{(12)}$ and $\Upsilon_{(21)}$ respectively. Therefore we can identify

$$\begin{aligned} M &\equiv n_r \\ M' &\equiv n_l \end{aligned}$$

and

$$\frac{N}{\mu r^2} = \hbar\omega_c.$$

Recall that we are using Planck units $\hbar, c = 1$, and the coupling constant is set to $q = \pm 1$. Thus using the definition of ω_c from (2.3) and $k \simeq \frac{2r^2}{N}$ from (3.2) for large N , we obtain

$$\frac{2}{k} = B \tag{4.26}$$

in the semi-classical limit. This is indeed precisely the field strength acting on strings Υ_{12} connecting the upper to the lower sheet near the poles as we have seen in (4.23). Therefore we have found complete agreement between the Landau problem and the string states on the squashed fuzzy sphere in the planar limit.

To illustrate this, we display in figure 3 and 4 the eigenfunctions of $\frac{1}{2\mu}\square_S$ for small and large N , and match these with the eigenstates of the Landau problem. The lowest Landau

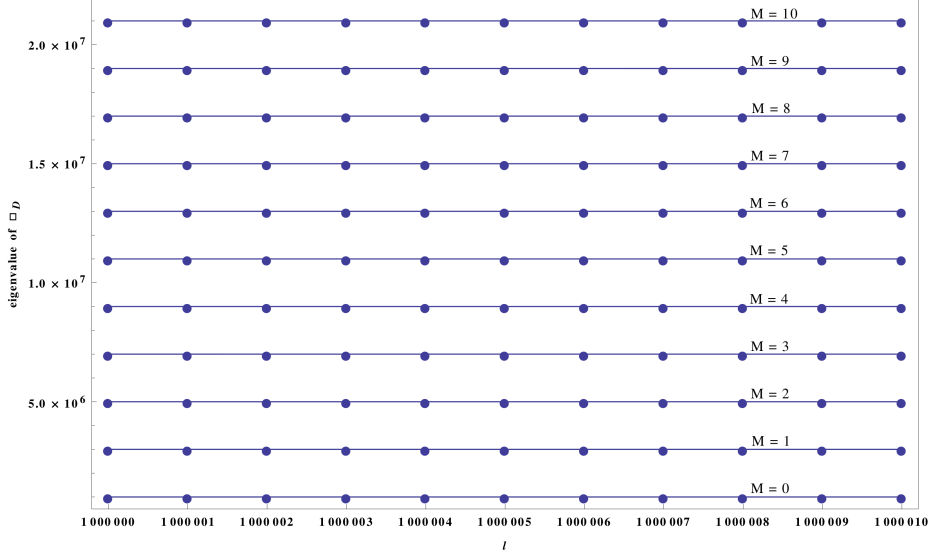


Figure 4: Levels $M = 0, \dots, 10$ for large l , if N is large. For large N these levels are approximately constant over this interval of l . Notice that the lowest level $M = 0$ is not 0, but has an offset, compatible with the result in equation (4.25).

level is given by the wave functions $\chi_{0,n}$, where n is either n_l or n_r depending on the sign of the \vec{B} -field. The lowest level of $\frac{1}{2\mu}\square_S$ is given by the \hat{Y}_l^l or \hat{Y}_{-l}^l for very large l . Thus the $\chi_{n_r,0}$ should be identified with \hat{Y}_{-l}^l , and the χ_{0,n_l} with \hat{Y}_l^l . Note that these are the highest and lowest weight states in the algebra of functions $Mat(N, \mathbb{C})$. In particular, $\chi_{n_r=0, n_l=0}$ can be identified with both $\hat{Y}_{l_{max}}^{l_{max}} = \Upsilon_{(12)}^{0,0}$ or $\hat{Y}_{-l_{max}}^{l_{max}} = \Upsilon_{(21)}^{0,0}$. The reason for this doubling is that we have both charged sectors $q = \pm 1$ realized at the same time, by the Υ_{12} and Υ_{21} respectively. We can thus identify the states in the various Landau levels as

$$\begin{aligned} \chi_{0,n_l} &\longleftrightarrow \hat{Y}_{l_{max}-n_l}^{l_{max}-n_l} = \Upsilon_{(12)}^{n_l,0}, \\ \chi_{1,n_l} &\longleftrightarrow \hat{Y}_{l_{max}-n_l-1}^{l_{max}-n_l-1} = \Upsilon_{(12)}^{n_l,1}, \\ \chi_{2,n_l} &\longleftrightarrow \hat{Y}_{l_{max}-n_l-2}^{l_{max}-n_l-2} = \Upsilon_{(12)}^{n_l,2} \end{aligned} \quad (4.27)$$

and so forth. Similarly for the opposite charges,

$$\begin{aligned} \chi_{n_r,0} &\longleftrightarrow \hat{Y}_{-(l_{max}-n_r)}^{l_{max}-n_r} = \Upsilon_{(21)}^{n_r,0}, \\ \chi_{n_r,1} &\longleftrightarrow \hat{Y}_{-(l_{max}-n_r)+1}^{l_{max}-n_r} = \Upsilon_{(21)}^{n_r,1} \end{aligned} \quad (4.28)$$

and so forth. The quantum number labeling the degenerate states in a Landau level can be identified using the operator $J_z^{(ad)} = \kappa^{-1}[X_3, \cdot]$, which corresponds to angular momentum around the z axis on the fuzzy sphere. We can compute its eigenvalue either directly from the $SU(2)$ quantum numbers

$$J_z^{(ad)}\Upsilon_{(12)}^{n_l,0} = (N - 1 - n_l), \quad J_z^{(ad)}\Upsilon_{(21)}^{n_r,0} = -(N - 1 - n_r) \quad (4.29)$$

or in the semi-classical limit $N \rightarrow \infty$ as follows

$$\begin{aligned}
J_z^{(\text{ad})} \Upsilon_{(12)}^{n_l,0} &= \kappa^{-1} (X_3 \Upsilon_{(12)} - \Upsilon_{(12)} X_3) \\
&= \sqrt{C_N} \left(1 - \frac{1}{2r^2} ((X^1)^2 + (X^2)^2) + O\left(\frac{x^4}{r^4}\right) \right) \Upsilon_{(12)} \\
&\quad - \sqrt{C_N} \Upsilon_{(12)} \left(-1 + \frac{1}{2r^2} ((X^1)^2 + (X^2)^2) + O\left(\frac{x^4}{r^4}\right) \right) \\
&\sim \frac{N}{2} \left(2 - \frac{1}{r^2} ((x^1)^2 + (x^2)^2) + O\left(\frac{x^4}{r^4}\right) \right) \Upsilon_{(12)}.
\end{aligned} \tag{4.30}$$

Neglecting the $O(\frac{x^4}{r^4})$ terms and taking into account the above identifications, we find

$$((x^1)^2 + (x^2)^2) \Upsilon_{(12)}^{n_l,0} = \frac{r^2}{\sqrt{C_N}} (n_l + 1) \Upsilon_{(12)}^{n_l,0} = \frac{2}{B} (n_l + 1) \Upsilon_{(12)}^{n_l,0}. \tag{4.31}$$

Hence these states are localized on circles around the origin with radius measured by n_l , just like the states χ_{0,n_l} in the Landau problem (2.5). The analogous statement for $\chi_{n_r,0}$ completes the identification of the harmonics on the squashed fuzzy sphere with those of the Landau problem.

Finally we can exhibit the stringy interpretation of these matrix states as links between the sheets. The states $\hat{Y}_{l_{max}}^{l_{max}} = \Upsilon_{(12)}^{0,0}$ or $\hat{Y}_{-l_{max}}^{l_{max}} = \Upsilon_{(21)}^{0,0}$ can be written explicitly as follows

$$\begin{aligned}
\Upsilon_{(12)}^{0,0} &= \left| \frac{N-1}{2} \right\rangle \left\langle -\frac{N-1}{2} \right| \\
\Upsilon_{(21)}^{0,0} &= \left| -\frac{N-1}{2} \right\rangle \left\langle \frac{N-1}{2} \right|.
\end{aligned} \tag{4.32}$$

Here the extremal weight states $|\pm \frac{N-1}{2}\rangle$ are the coherent states localized at the north and south pole of the fuzzy sphere, hence at the origin of the two fuzzy disks. This makes the interpretation of the $\Upsilon_{(12)}^{0,0}$ as strings connecting the two sheets at the origin manifest, and vindicates the identification with $\chi_{0,0}$ (4.27), (4.28). Although the expressions of the other states in terms of coherent states is more complicated, it is clear that they can be thought of as slightly extended strings localized at circles around the origin. This is illustrated in figure 5.

4.5 Dirac operator

For completeness, we briefly discuss also the Dirac operator on the squashed fuzzy sphere. \mathcal{D} is naturally defined by

$$\mathcal{D} = \frac{1}{\hbar} (\sigma_1 \otimes [X^1, \cdot] + \sigma_2 \otimes [X^2, \cdot])$$

with X^1, X^2 from (4.16). We can compute its square

$$\mathcal{D}^2 = \frac{1}{\hbar^2} \sigma_i \sigma_j \otimes [X^i, [X^j, \cdot]] \tag{4.33}$$

$$\begin{aligned}
&= \square_S \otimes \mathbb{1}_2 - \frac{1}{\hbar^2} [X^3, \cdot] \otimes \sigma_3 \\
&= \square \otimes \mathbb{1}_2 - \frac{1}{\hbar^2} ([X^3, \cdot] + \frac{1}{2} \sigma_3)^2 + \frac{1}{4\hbar^2}
\end{aligned} \tag{4.34}$$

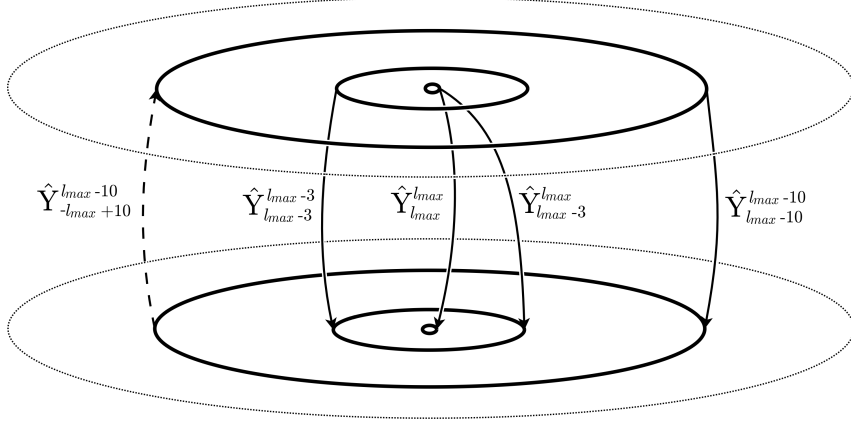


Figure 5: The strings Υ_{12} and Υ_{21} should be thought of as connecting the upper and lower sheet, creating transitions from one to the other. The solid arrows create transitions from the upper to the lower sheet and should be identified with Υ_{12} , while the dashed with Υ_{21} . Strings creating transitions from pole to pole will have very large quantum number $|m| \approx l_{max}$ in terms of $\hat{Y}_{\pm m}^{l_{max}}$.

using the commutation relations of the fuzzy sphere. The last form allows to find immediately the eigenvalues and eigenfunctions following [18]: Decomposing the space of functions $Mat(N, \mathbb{C}) = \oplus_{l=0}^{N-1} \mathbb{C}^{2l+1}$ with basis $|l, m_l\rangle$ and passing to the total angular momentum basis of $\mathbb{C}^2 \otimes \mathbb{C}^{2l+1}$ labeled by j, l, m_j , the eigenvalues are

$$E_{jlm_j}^2 = 4l(l+1) - 4m_j^2 + 1 = 4(l + \frac{1}{2} - m_j)(l + \frac{1}{2} + m_j) \quad (4.35)$$

where $m_j = m + s$, and s is the eigenvalue of $\frac{1}{2}\sigma_3$. Hence for each $l \in \{0, 1, 2, \dots, N-1\}$, there is pair of zero modes with extremal weights $m_j = \pm(l + \frac{1}{2})$, which can be written as

$$\Psi_+ = |\uparrow\rangle|l, l\rangle, \quad \Psi_- = |\downarrow\rangle|l, -l\rangle. \quad (4.36)$$

Thus the fermionic zero modes correspond to the extremal weight states in the angular momentum decomposition of $Mat(N, \mathbb{C})$. In particular for $L = 0$ or $l = l_{max}$, these can again be written in terms of coherent states as in (4.32). More generally, these zero modes can be interpreted as fermionic strings, linking the two opposite sheets at or near the origin.

On the other hand, the second form in (4.34) allows to easily take the semi-classical (pole) limit as in the previous section. Using the analogous procedure as for the Laplacian before, we obtain

$$\frac{1}{2\mu} \mathcal{D}^2 \Upsilon_{(12)}^{L,M} = \hbar\omega_c \left(M + \frac{1}{2} - \frac{\sigma_3}{2} \right) \Upsilon_{(12)}^{L,M}$$

in the large N limit using $\frac{2N}{r^2} = \hbar\omega_c$ for (4.26), and similarly for the $\Upsilon_{(21)}^{L,M'}$. Thus for these string states, $\frac{1}{2\mu} \mathcal{D}^2$ reproduces the Hamiltonian of the Landau levels including spin (2.6), for $g = 1$. In particular, we can understand the above fermionic zero modes as fermions in the lowest Landau level with appropriate orientation of the spin. Remarkably, they are exact zero modes even for finite N . For generalizations we refer the reader to [18].

5 Conclusion

We have identified string-like modes among the fuzzy spherical harmonics, which connect the upper with the lower hemisphere. On the squashed fuzzy sphere, these behave like charged objects moving under the influence of a magnetic field. In the large N limit, this field becomes approximately constant in the vicinity of the origin resp. the north and south poles, and the lowest string-like modes behave like charged point-like objects. In particular, we have identified the lowest Landau levels among these fuzzy spherical harmonics, providing an organization of the space of functions in terms of string-like modes.

Our results illustrate the well-known fact that noncommutative field theory is much richer than ordinary gauge theory, and behaves more like a string theory rather than a field theory. The present example provides a particularly clear identification of such string modes in a non-trivial background, in a simple finite-dimensional setting. It illustrates how non-trivial backgrounds in matrix models can be understood quantitatively in the semi-classical limit. The present example is related to the new solutions of (deformed) $\mathcal{N} = 4$ SYM and the IKKT matrix model [18, 26], which could be analyzed in a similar way. More generally, a systematic use of analogous string-like modes in the study of noncommutative field theory might help to illuminate various issues and problems in this context.

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